THE COLLEGES OF OXFORD UNIVERSITY

MATHEMATICS

Sample Solutions for *Issued May 2001* Written Test

1. A. If we substitute $x = y + t$ into the equation

$$0 = x^3 + ax^2 + bx + c,$$

the equation transforms into

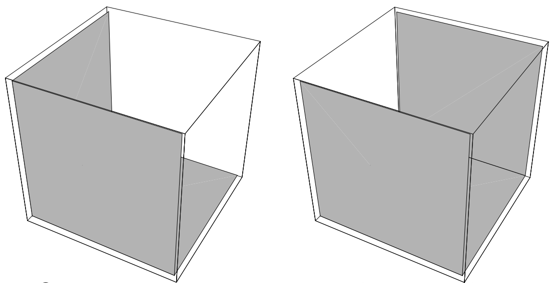
$$\begin{aligned} 0 &= (y+t)^3 + a(y+t)^2 + b(y+t) + c \\ &= y^3 + (3t+a)y^2 + \left(3t^2 + 2at + b\right)y + \left(t^3 + at^2 + bt + c\right), \end{aligned}$$

which has the required form $0 = y^3 + py + q$ if $3t + a = 0$. Hence we need $t = -a/3$ and the **answer is (b)**.

B. We are told that three faces of a cube are coloured red, and the other three coloured blue. As we can orient the cube however we please, we can assume that the uppermost face is red. If the bottom face is also red then exactly one of the side faces will also be red. In this case the three red faces form a C, vice-like around the side of the cube, while the three blue faces form a second C.

If the bottom face isn't red, then exactly two of the side faces must be red. These two faces are either adjacent or opposite. If they are opposite, then we are again in the situation of having two different coloured Cs around the cube. If they are adjacent, then the three red faces are all around one corner of the cube, with the three blue faces adjacent to the opposite corner.

So without knowing the orientation of the cube, all we can say is that the red and blue faces either form two clasping Cs, or they are the faces adjacent to two opposite vertices. There are two configurations and the **answer is (a)**.

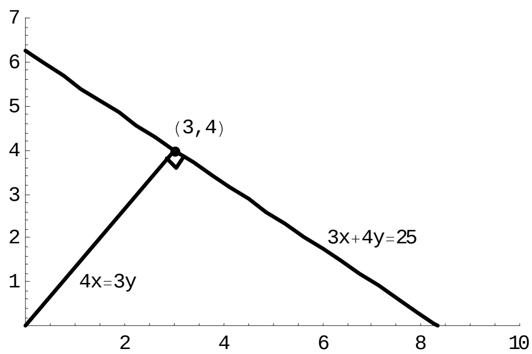


Opposite corners Clasping Cs

C. Consider the line $3x + 4y = 25$. It has gradient $-3/4$, and so the gradient of a line normal to it will have gradient $4/3$. The line through the origin with gradient $4/3$ is $3y = 4x$. If we solve these simultaneous equations we see that the two lines meet at $(3, 4)$. This is at distance

$$\sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

from the origin and so the **answer is (c)**.



D. The cubic equation

$$(x-\alpha)(x-\beta)(x-\gamma)=0$$

expands out to become

$$x^3-(\alpha+\beta+\gamma)\,x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)\,x-\alpha\beta\gamma=0.$$

So we see that the sum of the roots is negative the coefficient of the x^2 term, and the product of the roots is negative the constant coefficient.

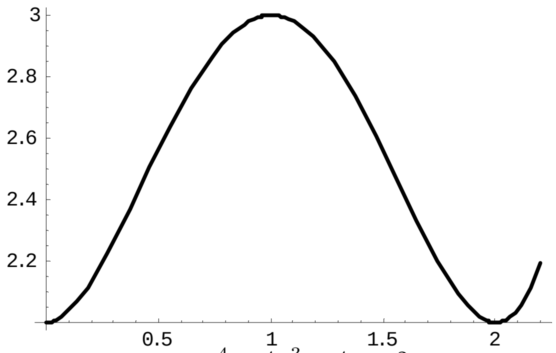
In our case the roots are $10, 11, -12$, and so

$$\begin{aligned}\text{coefficient of } x^2 &= -(10+11-12)=-9,\\ \text{constant coefficient} &= -(10\times 11\times -12)=1320,\end{aligned}$$

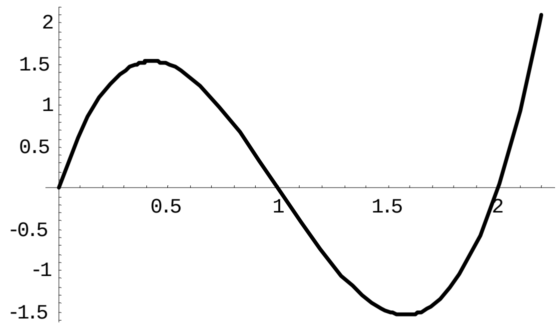
and of the given possibilities there is only one such and so the **answer is (c)**.

E. The gradient (i.e. the derivative) of the curve $y=x^4-4x^3+4x^2+2$ is

$$\text{gradient}=\frac{\mathrm{d}y}{\mathrm{d}x}=4x^3-12x^2+8x=4x\,(x-1)\,(x-2)\,.$$



$y=x^4-4x^3+4x+2$



$y=4x\,(x-1)\,(x-2)$

In the interval $0\le x\le 2\frac{1}{5}$ the gradient will attain its maximum either at an endpoint (0 or $2\frac{1}{5}$), or at a maximum turning point. Note

$$\begin{aligned}\text{gradient}\,(0) &= 0,\\ \text{gradient}\left(2\frac{1}{5}\right) &= 4\times\frac{11}{5}\times\frac{6}{5}\times\frac{1}{5}=\frac{264}{125}>2.\end{aligned}$$

As the gradient is a cubic in x, with a positive leading coefficient and roots $0,1,2$, then from our knowledge of the graph of a cubic, we see the gradient will have a maximum turning point between $x=0$ and $x=1$, and a minimum turning point between $x=1$ and $x=2$. To find these points we need to differentiate the gradient — so

$$\frac{\mathrm{d}\,(\text{gradient})}{\mathrm{d}x}=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=12x^2-24x+8=12\left((x-1)^2-\frac{1}{3}\right),$$

which we see equals zero at $1+\frac{1}{\sqrt{3}}$ and $1-\frac{1}{\sqrt{3}}$. Our maximum turning point is then at $x=1-\frac{1}{\sqrt{3}}$ and the corresponding gradient is

$$\text{gradient}\left(1-\frac{1}{\sqrt{3}}\right)=4\left(1-\frac{1}{\sqrt{3}}\right)\left(\frac{-1}{\sqrt{3}}\right)\left(-1-\frac{1}{\sqrt{3}}\right)=\frac{8}{3\sqrt{3}}<2.$$

Finally, having considered all the possibilities, we see that the maximum gradient is at $x=2\frac{1}{5}$. The **answer is (d).**

F. The expression

$$x^2y+xy^2+y^2z+yz^2+z^2x+zx^2-x^3-y^3-z^3-2xyz$$

is symmetric in the variables x, y and z. That is, if we replace x, y, z with some rearrangement of x, y, z (say we replace x with y, y with z and z with x) then the expression will remain the same. As this is the case then this will also be true of any factorisation of the expression.

Now factoristion (a) isn't symmetric because it contains the bracket $(x-y+z)$, but not the factor $(x+y-z)$ — from a swap of y and z. Similarly, factorisation (b) isn't symmetric, as it contains the factor $(x+y-z)$, but not the factor $(x-y+z)$. Factorisations (c) and (d) *are* symmetric, (they are negative one another), but if we look for the coefficients of the x^3 term in each of these we see that it is -1 in (c) and 1 in (d). Hence the **answer is (c)**.

G. By the product rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(uvw) &= \frac{\mathrm{d}u}{\mathrm{d}x}vw + u\frac{\mathrm{d}}{\mathrm{d}x}(vw)\\ &= \frac{\mathrm{d}u}{\mathrm{d}x}vw + u\frac{\mathrm{d}v}{\mathrm{d}x}w + uv\frac{\mathrm{d}w}{\mathrm{d}x},\end{aligned}$$

and the chain rule, we see

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(xe^{-x^2}\cos\left(\frac{1}{x}\right)\right) &= 1\times e^{-x^2}\times\cos\left(\frac{1}{x}\right) + x\times -2xe^{-x^2}\times\cos\left(\frac{1}{x}\right) + x\times e^{-x^2}\times\frac{-1}{x^2}(-\sin)\left(\frac{1}{x}\right)\\ &= e^{-x^2}\cos\left(\frac{1}{x}\right) - 2x^2e^{-x^2}\cos\left(\frac{1}{x}\right) + \frac{1}{x}e^{-x^2}\sin\left(\frac{1}{x}\right),\end{aligned}$$

and so we see that the **answer is (b)**.

H. We are told the sums of two infinite sums, namely

$$\begin{aligned}1^2 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots &= \frac{\pi^2}{6},\\ 1^2 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots &= \frac{\pi^2}{8}.\end{aligned}$$

So

$$\begin{aligned}\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots &= \left(1^2 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) - \left(1^2 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)\\ &= \frac{\pi^2}{6} - \frac{\pi^2}{8} = \frac{\pi^2}{24},\end{aligned}$$

and

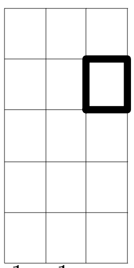
$$\begin{aligned}1^2 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots &= \left(1^2 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right)\\ &= \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12},\end{aligned}$$

and we see the **answer is (c).** (It is possible to answer this question only knowing the first sum — how would you do this?)

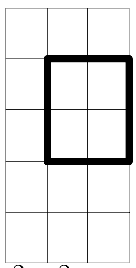
I. Using the grid, which has sides 3 and 5, we may draw squares that are 1 by 1, or 2 by 2, or 3 by 3. There are $3\times 5 = 15$ unit squares, $2\times 4 = 8$ squares of side 2, and $1\times 3 = 3$ squares of side 3 and so the total number of squares is

$$15 + 8 + 3 = 26,$$

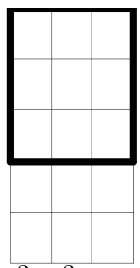
and we see the **answer is (c)**.



A 1×1 square



A 2×2 square



A 3×3 square

J. We shall call the four players in this game A, B, C, D, and call the two identical cards a and b. There are many such ways that two cards could appear in the same hand. One such way is that cards a and b are respectively the first and second cards in player A's hand — the chance of this happening equals

$$\text{(probability that } A\text{'s first card is } a) \times \text{(probability that } A\text{'s second card is } b \text{ given that } A\text{'s first card is } a) = \frac{1}{52} \times \frac{1}{51}$$

as there are 52 cards in the deck, and because, once we know the first card to be a, then the second card can be one of the remaining 51 cards.

All these many different ways are equally likely — so to find the total probability we need only count the number of different ways and multiply by the above probability. We note

$$\text{total number of different ways cards } a \text{ and } b \text{ appear in the same hand}$$

equal

$$4 \times \text{total number of different ways cards } a \text{ and } b \text{ appear player } A\text{'s hand,}$$

as the four players are all equally likely to get both cards. Then the above equals

$$4 \times \text{number of places } a \text{ can take in } A\text{'s hand } \times \text{ number of places } b \text{ can take in } A\text{'s hand given } a \text{ is already there,}$$

which equals $4 \times 13 \times 12$, because there are 13 cards in A's hand, and once one of these has been chosen to be a there are 12 remaining cards which can be card b.

Finally then we see the total probability is

$$\frac{4 \times 13 \times 12}{52 \times 51} = \frac{12}{51}$$

and we see that the **answer is (b).**

2. (a) By inspection $x^2 + 3x - 10 = (x+5)(x-2)$. (Of course you could use the quadratic formula if you failed to spot the roots.)

(b) We are told that for all values of x,

$$\begin{aligned} x^3 + ax^2 + bx + c &= (x-\alpha)(x-\beta)(x-\gamma) \\ &= \left(x^2 - (\alpha+\beta)x + \alpha\beta\right)(x-\gamma) \\ &= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma. \end{aligned}$$

Because this is an identity we can compare the coefficients of the powers of x and we see

$$\begin{aligned} a &= -(\alpha+\beta+\gamma) \\ b &= (\alpha\beta+\beta\gamma+\gamma\alpha) \\ c &= -\alpha\beta\gamma. \end{aligned}$$

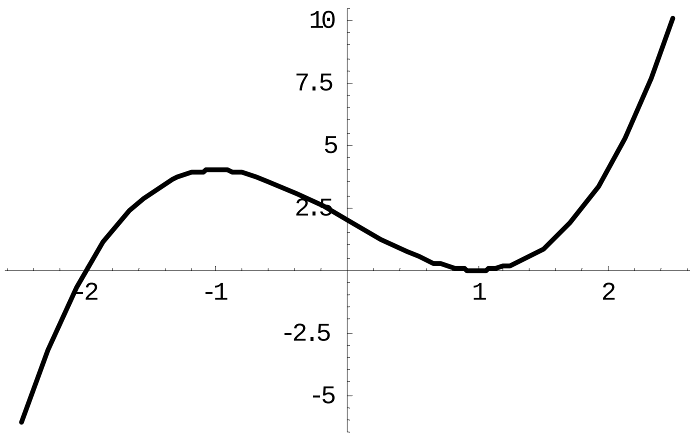
(c) We are told that the equation $x^3 + bx + 2 = 0$ has exactly two distinct roots — i.e. one of the roots is repeated. Let's call the repeated root α and the other root β. Looking at our fomula from part (b) above, we see that

$$\begin{aligned} -(2\alpha+\beta) &= 0 \\ \alpha^2 + 2\alpha\beta &= b \\ -\alpha^2\beta &= 2. \end{aligned}$$

If we substitute $\beta = -2\alpha$ from the first equation into the second two, we see that

$$-3\alpha^2 = b \text{ and } 2\alpha^3 = 2,$$

and we see that these have a solution when $\alpha = 1$ and $b = -3$. So the required value of b is -3.



$y = x^3 - 3x + 2 = (x-1)^2(x+2)$

3. (a) Consider the two curves $y = 6x^2$ and $y = x^4 - 16$. To find their intersections we can eliminate y and see that x must satisfy

$$x^4 - 16 = 6x^2.$$

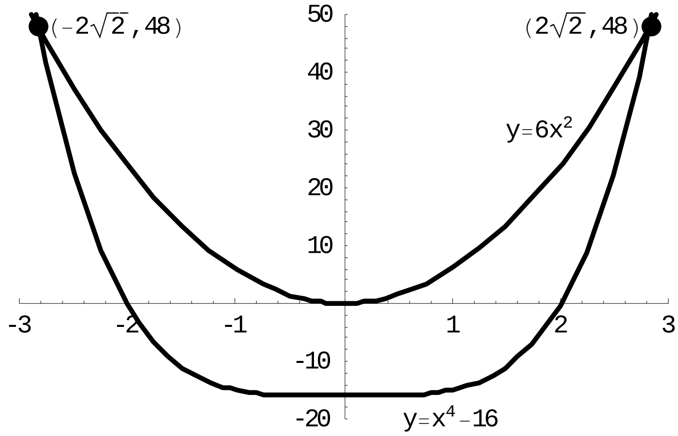
Rearranging we see that

$$0 = x^4 - 6x^2 - 16 = \left(x^2 - 8\right)\left(x^2 + 2\right).$$

The only real solutions of this equation are $x = 2\sqrt{2}$ and $x = -2\sqrt{2}$. Their respective y values are both 48 and so the two points of intersection are

$$\left(-2\sqrt{2}, 48\right) \quad \text{and} \quad \left(2\sqrt{2}, 48\right).$$

(b) A sketch of the two graphs should look something like:



(c) The region enclosed by the two curves lies in the range $-2\sqrt{2} \le x \le 2\sqrt{2}$, and the height of the region in this range is $6x^2 - \left(x^4 - 16\right)$ for each value of x in this range. So the enclosed area is given by the integral

$$\begin{aligned}
\int_{-2\sqrt{2}}^{2\sqrt{2}} \left(6x^2 - x^4 + 16\right)\, dx &= \left[2x^3 - \frac{x^5}{5} + 16x\right]_{-2\sqrt{2}}^{2\sqrt{2}} \\
&= 2\left(2\left(2\sqrt{2}\right)^3 - \frac{\left(2\sqrt{2}\right)^5}{5} + 16\left(2\sqrt{2}\right)\right) \\
&= 2\left(2 \times 16\sqrt{2} - \frac{128\sqrt{2}}{5} + 32\sqrt{2}\right) \\
&= \frac{384\sqrt{2}}{5}.
\end{aligned}$$

4. (a) The line $y = mx + c$ passes through the point $(1, 1)$ if $1 = m + c$, i.e. if $c = 1 - m$.

(b) We are told that L is a line through the point $(1, 1)$ which has positive gradient $m > 0$. So the equation of L is

$$y = mx + (1 - m)\,.$$

Now the line L', we are told, is perpendicular to L and hence the gradient of L' is $-1/m$. The equation of L' is then of the form

$$y = \frac{-1}{m}x + c'$$

for some value c'. As we are also told that L' also passes through the point $(1, a)$ (where $a \neq 1$), then we see that

$$a = \frac{-1}{m} + c' \quad \text{and so} \quad c' = a + \frac{1}{m}.$$

So the two equations of our lines L and L' are respectively

$$y = mx + (1 - m) \quad \text{and} \quad y = \frac{-x}{m} + \left(a + \frac{1}{m}\right).$$

(c) To find their intersection we can eliminate y and we have

$$mx + (1 - m) = \frac{-x}{m} + \left(a + \frac{1}{m}\right),$$

which becomes

$$\left(m + \frac{1}{m}\right) x = m + \frac{1}{m} + a - 1,$$

and hence

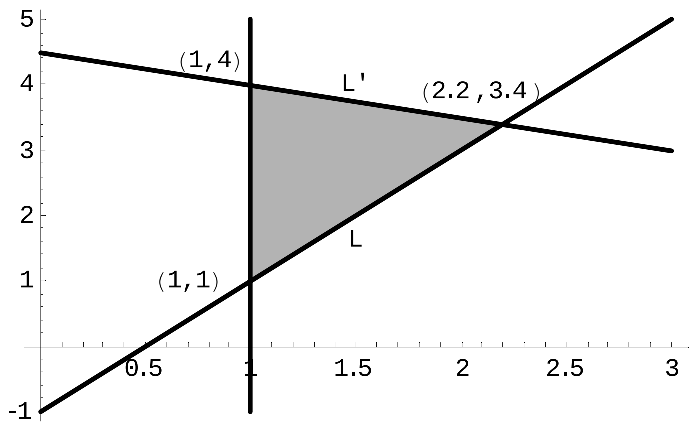
$$x = 1 + \frac{m\,(a - 1)}{m^2 + 1}.$$

We are asked for the area of the triangle which has this point of intersection as a vertex, and $(1, 1)$ and $(1, a)$ as the other two. If we think of the line segment between $(1, 1)$ and $(1, a)$ as the base of the triangle then the height of the triangle is

$$\left|\left(1 + \frac{m\,(a - 1)}{m^2 + 1}\right) - 1\right| = \frac{m\,|a - 1|}{m^2 + 1}.$$

Hence the area of the triangle is

$$\begin{aligned} \frac{1}{2} \times \text{base} \times \text{height} &= \frac{1}{2} \times |a - 1| \times \frac{m\,|a - 1|}{m^2 + 1} \\ &= \frac{m\,(a - 1)^2}{2\,(m^2 + 1)}. \end{aligned}$$



A sketch with $m = 2$ and $a = 4$

(d) As the angle at the intersection of L and L' is a right angle, then the triangle can only be isosceles by the other two angles being $\pi/4$. This happens when the gradient of L is $m = 1$.

5. (a) If we choose co-ordinates for the cube's corners then we can assume

$$A = (0,0,0) \quad \text{and} \quad B = (1,1,0)\,.$$

By walking along a single rod, the ant will change precisely one of its co-ordinates by 1. So we can see that at least two rods must be used in any path because the co-ordinates of A and B differ in two variables. Such paths of length two clearly exist such as

$$\begin{aligned} A &= (0,0,0) \to (1,0,0) \to (1,1,0) = B, \text{ and} \\ A &= (0,0,0) \to (0,1,0) \to (1,1,0) = B, \end{aligned}$$

and these are the only such paths of length two, as the ant has only two moves with which to change the x and y co-ordinates.

(b) and (c) Firstly we note that all paths must be of even length: to see this, note that after an odd/even number of moves, the sum of the co-ordinates of the point the ant is currently at will be odd/even. As B's co-ordinates add up to two then an even number of moves must be needed.

We now introduce some notation to simplify our descriptions of paths. We shall use the letters x, y and z to respectively denote a move that changes the co-ordinates of the ant in the x, y or z direction. So for example, the two shortest paths above would be denoted as xy and yx. Similarly $zxzy$ would denote the path

$$A = (0,0,0) \to (0,0,1) \to (1,0,1) \to (1,0,0) \to (1,1,0) = B.$$

We see the paths of length four are denoted by words which contain two zs, one x and one y. This is because it is impossible to have a path of length four in the $z = 0$ face without repeating a corner and if the ant moves onto the $z = 1$ face it must later move back. Further the two zs cannot be consecutive, as this would also result in a repeated vertex. So the paths of length four are

$$zxzy, zyzx, zxyz, zyxz, xzyz, yzxz,$$

and we see there are six such paths.

Consider now the paths of length six. Again there must be an even number of zs to get onto and away from the $z = 1$ face. There must be at least two, but there cannot be four, or else the ant would repeat a point because two of these zs would again be consecutive — so there are exactly two zs in these paths. Further, there must be an odd number of xs and an odd number of ys if, overall, the x and y co-ordinates are each to change from 0 to 1. So we will need three xs and one y, or vice-versa. Again we will need to avoid consecutive moves of the same type. Finally note that the word cannot begin xy, or yx, as this would mean that the ant arrives at B after only just two moves. Similarly they cannot end xy or yx, as this would mean that the ant had returned to A two moves before the end. To place three xs (say) in a non-consecutive fashion then they have to be put as

$$x * x * x * \ \text{ or } \ * x * x * x \text{ or } x * * x * x \text{ or } x * x * * x.$$

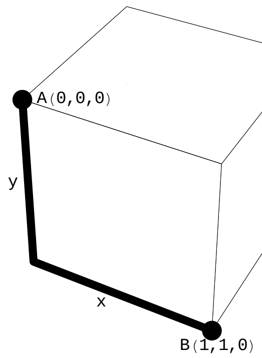
Let's look at the *potential* paths of length six which involve three xs. These are

$$xzxyxz, zxyxzx, xzyxzx, xzxyzx.$$

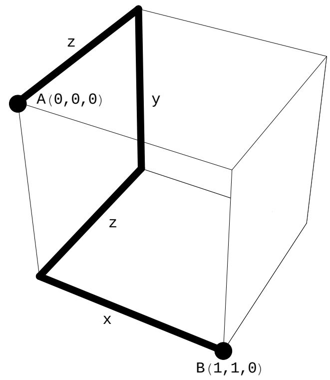
It is then easy to check each path out as having no repeats. There are also the four corresponding paths that involve three ys and one x — in total then there are eight paths of length six.

Are there paths of longer length? It is easy to see that there can be no paths of length greater than eight, because only one of the three rods connected to A and only one of the three at B can be used by the ant without causing a repeat. This means that at most eight of the rods could possibly be used in a path with no repeats. But if we also note that the ant could not use all four of the edges that lie in the $z = 1$ plane, without again causing a repeat, then we see that a path of length eight with no repeats is also impossible.

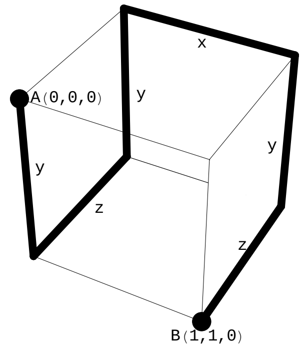
Here are sketches of some paths of lengths two, four and six.



The path yx



The path $zyzx$



The path $yzyxyz$

N.B. This level of analysis in parts (b) and (c) was not expected in answer to this question when it was set as part of the Entrance Test. The answer has been presented this way to show how the question can be systematically answered. For the Test it would have been perfectly satisfactory to count up the number of paths of each length and to present an answer without any further justification.

(d) Besides A and B there are six other vertices on the cube. If we had a path that began at A, passed through these six vertices exactly twice, and then moved to B we would have a path that (counting repetitions) had passed through fourteen vertices. We have already noted that a path from A to B uses an even number of rods and hence passes through an odd number of vertices — so no such path is possible.

(These sample solutions have been produced by Dr. Richard Earl, who is the Schools Liaison and Access for mathematics, statistics and computer science in Oxford. Any questions regarding the test, applying to Oxford to read for these subjects, or comments on these solutions, would be welcome at earl@maths.ox.ac.uk)